Correction of the exercises from the book

*A Wavelet Tour of Signal Processing*

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Abstract

These corrections refer to the 3rd edition of the book *A Wavelet Tour of Signal Processing – The Sparse Way* by Stéphane Mallat, published in December 2008 by Elsevier. If you find mistakes or imprecisions in these corrections, please send an email to Gabriel Peyré (gabriel.peyre@ceremade.dauphine.fr). More information about the book, including how to order it, numerical simulations, and much more, can be find online at wavelet-tour.com.

1 Chapter 2

**Exercise 2.1.** For all $t$, the function $\omega \mapsto e^{-i\omega t}f(t)$ is continuous. If $f \in L^1(\mathbb{R})$, then for all $\omega$, $|e^{-i\omega t}f(t)| \leq |f(t)|$ which is integrable. One can thus apply the theorem of continuity under the integral sign $\hat{f}$ which proves that $\hat{f}$ is continuous.

If $\hat{f} \in L^1(\mathbb{R})$, using the inverse Fourier formula (2.8) and a similar argument, one proves that $f$ is continuous.

**Exercise 2.2.** If $\int |h| = +\infty$, for all $A > 0$ there exists $B > 0$ such that $\int_{-B}^{B} |h| > A$. Taking $f(x) = 1_{[-A,A]} \text{sign}(h(-x))$ which is integrable and bounded by 1 shows that

$$f \ast h(0) = \int_{-B}^{B} \text{sign}(h(t))h(t)dt > A.$$ 

This shows that the operator $f \mapsto f \ast h$ is not bounded on $L^{\infty}$, and thus the filter $h$ is unstable.

**Exercise 2.3.** Let $f_u(t) = f(t - u)$, by change of variable $t - u \to t$, one gets

$$\hat{f}_u(\omega) = \int f(t - u)e^{-i\omega t}dt = \int f(t)e^{-i\omega(t+u)}dt = e^{-i\omega u}\hat{f}(\omega).$$

Let $f_s(t) = f(t/s)$, with $s > 0$, by change of variable $t/s \to t$, one get

$$\hat{f}_s(\omega) = \int f(t/s)e^{-i\omega t}dt = \int f(t)e^{-i\omega st}|s|dt = |s|\hat{f}(s\omega).$$
Let $f$ by $C^1$ and $g = f'$, the by integration by parts, since $f(t) \to 0$ where $|t| \to +\infty$,

$$
g(\omega) = \int f'(t)e^{-i\omega t}dt = -\int f(t)(-i\omega)e^{-i\omega t}dt = (i\omega)f'(\omega).
$$

**Exercise 2.4.** One has

$$
f_r(t) = \text{Re}[f(t)] = [f(t) + f^*(t)]/2 \quad \text{and} \quad f_i(t) = \text{Im}[f(t)] = [f(t) - f^*(t)]/2
$$

so that

$$
\hat{f}_r(\omega) = \int \frac{f(t) + f^*(t)}{2}e^{-i\omega t}dt = \frac{\hat{f}(\omega)}{2} + \text{Conj} \left( \int f(t)e^{i\omega t}dt \right)/2
$$

$$
= [\hat{f}(\omega) + \hat{f}^*(-\omega)]/2,
$$

where $\text{Conj}(a) = a^*$ is the complex conjugate. The same computation leads to

$$
\hat{f}_i(\omega) = [\hat{f}(\omega) - \hat{f}^*(-\omega)]/2.
$$

**Exercise 2.5.** One has

$$
\hat{f}(0) = \int f(t)dt = 0.
$$

If $f \in L^1(\mathbb{R})$, one can apply the theorem of derivation under the integral sign $\int$ and get

$$
\frac{d}{d\omega}\hat{f}(\omega) = \int -itf(t)e^{-i\omega t}dt \quad \Rightarrow \quad \hat{f}'(0) = -i\int tf(t)dt = 0.
$$

**Exercise 2.6.** If $f = 1_{[-\pi,\pi]}$ then one can verify that

$$
\hat{f}(\omega) = \frac{2\sin(\pi\omega)}{\omega}.
$$

It result that

$$
\int \frac{\sin(\pi\omega)}{\pi\omega} d\omega = \frac{1}{2\pi} \int \hat{f}(\omega)d\omega = f(0) = 1.
$$

If $g = 1_{[-1,1]}$ then $\hat{g}(\omega)/2 = \sin(\omega)/\omega$. The inverse Fourier transform of $\hat{g}(\omega)^3$ is $g * g * g(t)$ so

$$
\int \frac{\sin^3(\omega)}{\omega^3} d\omega = \frac{1}{8} \int \hat{g}(\omega)^3 d\omega = \frac{2\pi}{8} g * g * g(0) = \frac{3\pi}{4},
$$

where we used the fact that

$$
g * g * g(0) = \int_{-1}^{1} h(t)dt = 3
$$

where $h$ is a piecewise linear hat function with $h(0) = 2$.

**Exercise 2.7.** Writing $u = a - ib$, and differentiating under the integral sign $\int$, one has

$$
f'(\omega) = \int -ite^{-iut}e^{-i\omega t}dt.
$$

By integration by parts, one gets an ordinary differential equation

$$
f'(\omega) = \frac{-\omega}{2u} \hat{f}(\omega)
$$
whose solution is
\[ f(\omega) = Ke^{-\frac{\omega^2}{\pi}} \]
for some constant \( K = \hat{f}(0) \). Using a switch from Euclidean coordinates to polar coordinates \((x,y) \rightarrow (r,\theta)\) which satisfies \(dxdy = rdr\theta\), one gets
\[
K^2 = \iint e^{-ux^2}e^{-uy^2}dxdy = \iiint e^{-u(x^2+y^2)}dxdy
\]
\[
= \int_0^{2\pi} \int_0^{+\infty} e^{-ur^2}rdrd\theta = 2\pi \int_0^{+\infty} re^{-ur^2}dr = \frac{\pi}{u},
\]
which gives the result.

**Exercise 2.8.** If \( f \) is \( C^1 \) with a compact support, with an integration by parts we get
\[
\hat{f}(\omega) = \frac{1}{i\omega} \int f'(t)e^{-i\omega t}dt
\]
so that
\[
|\hat{f}(\omega)| \leqslant \frac{C}{\omega} \quad \text{with} \quad C = \int |f'(t)|dt < +\infty,
\]
which proves that \( f(\omega) \rightarrow 0 \) when \( |\omega| \rightarrow +\infty \).

Let \( f \in L^1(\mathbb{R}) \) and \( \varepsilon > 0 \). Since \( C^1 \) functions are dense in \( L^1(\mathbb{R}) \), one can find \( g \) such that
\[
\int |f-g| \leqslant \varepsilon/2.
\]
Since \( \hat{g}(\omega) \rightarrow 0 \) when \( |\omega| \rightarrow +\infty \), there exists \( A \) such that \( |\hat{g}(\omega)| \leqslant \varepsilon/2 \) when \( |\omega| > A \). Moreover, the Fourier integral definition implies that
\[
|\hat{f}(\omega) - \hat{g}(\omega)| \leqslant \int |f(t) - g(t)| dt
\]
so for all \( |\omega| > A \) we have \( |\hat{f}(\omega)| \leqslant \varepsilon \) which proves that \( f(\omega) \rightarrow 0 \) when \( |\omega| \rightarrow +\infty \).

**Exercise 2.9. a)** For \( f_0(t) = 1_{[0,\infty)}(t)e^{pt} \), we get
\[
\hat{f}_0(\omega) = \int_0^{+\infty} e^{(p-i\omega)t}dt = \frac{1}{i\omega - p}.
\]

For \( f_n(t) = t^n1_{[0,\infty)}(t)e^{pt} \), an integration by parts gives
\[
\hat{f}_n(\omega) = \int_0^{+\infty} t^n e^{(p-i\omega)t}dt = \frac{n}{i\omega - p}\hat{f}_{n-1}(\omega),
\]
so that
\[
\hat{f}_n(\omega) = \frac{n!}{(i\omega - p)^n}.
\]

**b)** Computing the Fourier transform on both sides of the differential equation gives
\[ g = f \ast h \quad \text{where} \quad \hat{h}(\omega) = \frac{\sum_{k=0}^K a_k(i\omega)^k}{\sum_{k=0}^M b_k(i\omega)^k}. \]

We denote by \( \{p_k\}_{k=0}^L \) the poles of the polynomial \( \sum_{k=0}^M b_kz^k \), with multiplicity \( n_k \). If \( K < M \),
one can decompose the rational fraction into
\[
\hat{h}(\omega) = \sum_{k=0}^L \frac{Q_k(i\omega)}{(i\omega - p_k)^{n_k}}
\]
where each $Q_k$ is a polynomial of degree strictly smaller than $n_k$. It results that $h(t)$ is a sum of derivatives up to a degree strictly smaller than $n_k$ of the inverse Fourier transform of

$$\hat{f}_{p_k,n_k}(\omega) = \frac{1}{(i\omega - p_k)^{n_k}}$$

which is

$$f_{p_k,n_k}(t) = \frac{1}{n_k!} t^{n_k} 1_{[0,\infty)}(t) e^{p_k t}.$$ 

Each filter $f_{p_k,n_k}$ is causal, stable and $n_k$ times differentiable. It results that that $h$ is causal and stable.

If, there exists $l$ with $\text{Re}(p_l) = 0$ then for the frequency $\omega = -ip_l$ we have $|\hat{h}(\omega)| = +\infty$ so $h$ cannot be stable.

If, there exists $l$ with $\text{Re}(p_l) > 0$ then by observing that $\hat{f}_{p_l,n_l}(-\omega) = (-1)^{n_l} (i\omega + p_l)^{-n_l}$ and by applying the result in a) we get

$$f_{p_l,n_l}(t) = \frac{1}{n_l!} t^{n_l} 1_{(-\infty,0)}(t) e^{-p_l t}$$

which is anticausal. We thus derive that $h$ is not causal.

c) Denoting $\alpha = e^{i\pi/3}$, one can write

$$|\hat{h}(\omega)|^2 = \frac{1}{1 - (i\omega/\omega_0)^b}$$

with

$$1/\hat{h}(\omega) = (i\omega/\omega_0 + 1)(i\omega/\omega_0 + \alpha)(i\omega/\omega_0 + \alpha^*) = P(i\omega).$$

Since the zeros of $P(z)$ have all a strictly negative real part, $h$ is stable and causal. To compute $h(t)$ we decompose

$$\hat{h}(\omega) = \frac{a_1}{i\omega/\omega_0 + 1} + \frac{a_2}{i\omega/\omega_0 + \alpha} + \frac{a_3}{i\omega/\omega_0 + \alpha^*},$$

we compute $a_1$, $a_2$ and $a_3$ and by applying the result in (a) we derive that

$$\hat{h}(t) = \omega_0(1_{[0,\infty)}(t) e^{-t\omega_0} + a_2 1_{[0,\infty)}(t) e^{-\alpha t\omega_0} + a_3 1_{[0,\infty)}(t) e^{-\alpha^* t\omega_0}) .$$

**Exercise 2.10.** For $a > 0$ and $u > 0$ and $g$ a Gaussian function, define

$$f_{a,u}(t) = e^{iat} g(t - u) + e^{-iat} g(t + u).$$

We verify that $\sigma_w(f_{a,u})$ increases proportionally to $u$. Its Fourier transform is

$$\hat{f}_{a,u}(\omega) = e^{-iua} \hat{g}(\omega - a) + e^{iua} \hat{g}(\omega + a)$$

so $\sigma_w(f_{a,u})$ increases proportionally to $a$. For $a$ and $u$ sufficiently large we get the the result.

**Exercise 2.11.** Since $f(t) \geq 0$

$$|\hat{f}(\omega)| = \left| \int f(t) e^{-i\omega t} dt \right| \leq \int f(t) dt = \hat{f}(0) .$$

**Exercise 2.12.** a) Denoting $u(t) = |\sin(t)|$, one has $g(t) = a(t) u(\omega_0 t)$ so that

$$\hat{g}(\omega) = \frac{1}{2\pi} \hat{a}(\omega) * \hat{u}(\omega/\omega_0)$$
where \( \hat{u}(\omega) \) is a distribution
\[
\hat{u}(\omega) = \sum_n c_n \delta(\omega - n)
\]
and \( c_n \) is the Fourier coefficient
\[
c_n = \int_{-\pi}^{\pi} |\sin(t)| e^{-int} \, dt = -\int_{-\pi}^{0} \sin(t) e^{-int} \, dt + \int_{0}^{\pi} \sin(t) e^{-int} \, dt.
\]
The change of variable \( t \to t + \pi \) in the first integral shows that \( c_{2k+1} = 0 \) and for \( n = 2k \),
\[
c_{2k} = 2 \int_{0}^{\pi} \sin(t) e^{-i2kt} \, dt = \frac{4}{1 - 4k^2}.
\]
One thus has
\[
\hat{u}(\omega) = \frac{1}{2\pi} \sum_n c_n \hat{u}(\omega - n\omega_0) = \frac{2}{\pi} \sum_k \hat{a}(\omega - 2k\omega_0) - \frac{1}{1 - 4k^2}.
\]

b) If \( \hat{a}(\omega) = 0 \) for \( |\omega| > \omega_0 \), then \( h \) defined by \( \hat{h}(\omega) = \frac{\pi}{2} 1_{[-\omega_0,\omega_0]} \) guarantees that \( \hat{g} \hat{h} = \hat{a} \) and hence \( a = g \ast h \).

**Exercise 2.13.** One has
\[
\hat{g}(\omega) = \frac{1}{2} \sum_n \hat{f}_n(\omega) \ast [\delta(\omega - 2n\omega_0) + \delta(\omega + 2n\omega_0)] = \frac{1}{2} \sum_n [\hat{f}_n(\omega - 2n\omega_0) + \hat{f}_n(\omega + 2n\omega_0)].
\]
Each \( \hat{f}_n(\omega \pm 2n\omega_0) \) is supported in \( [(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0] \), and thus \( \hat{g} \) is supported in \( [-2N\omega_0, 2N\omega_0] \).

Since the intervals \( [(-1 \pm 2n)\omega_0, (1 \pm 2n)\omega_0] \) are disjoint, one has
\[
\hat{f}_n(\omega \pm 2n\omega_0) = 2\hat{g}(\omega)1_{[-1 \pm 2n\omega_0, 1 \pm 2n\omega_0]}(\omega).
\]
The change of variable \( \omega \pm 2n\omega_0 \to \omega \) and summing for \( n \) and \(-n\) gives
\[
\hat{f}_n(\omega) = [\hat{g}(\omega - 2n\omega_0) + \hat{g}(\omega + 2n\omega_0)]\hat{h}(\omega),
\]
where \( \hat{h}(\omega) = 1_{[-\omega_0, \omega_0]}(\omega) \). Denoting \( g_n(t) = 2g(t) \cos(2n\omega_0 t) \), one sees that \( f_n \) is recovered as
\[
f_n = g_n \ast h.
\]

**Exercise 2.14.** The function \( \phi(t) = \sin(\pi t)/(\pi t) \) is monotone on \([-3/2, 0]\) and \([0, 3/2]\) on which is variation is \( 1 + \tfrac{1}{\pi^2} \). For each \( k \in \mathbb{N}^* \), it is also monotone on each interval \([k + 1/2, k + 3/2]\) on which the variation is \( \frac{1}{\pi}[(k + 1/2)^{-1} + (k + 3/2)^{-1}] \). One thus has
\[
\|\phi\|_V = 2(1 + \frac{2}{3\pi}) + \frac{2}{\pi} \sum_{k \geq 1} [(k + 1/2)^{-1} + (k + 3/2)^{-1}] = +\infty.
\]
For \( \phi = \lambda 1_{[a, b]} \), \( \phi' = \lambda \delta_a + \lambda \delta_b \) and hence \( \|\phi\|_V = 2\lambda \).

**Exercise 2.16.** Let
\[
f(x) = 1_{[0,1]^2}(x_1, x_2) = f_0(x_1)f_0(x_2) \quad \text{where} \quad f_0(x_1) = 1_{[0,1]}(x_1).
\]
One has
\[ \hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1) \hat{f}_0(\omega_2) = \frac{(e^{i\omega_1} - 1)(e^{i\omega_2} - 1)}{\omega_1 \omega_2}. \]

Let
\[ f(x) = e^{-x_1^2 - x_2^2} = f_0(x_1) f_0(x_2) \]
where \( f_0(x_1) = e^{-x_1^2} \).

One has
\[ \hat{f}(\omega_1, \omega_2) = \hat{f}_0(\omega_1) \hat{f}_0(\omega_2) = \pi e^{-(\omega_1^2 + \omega_2^2)/4}. \]

**Exercise 2.17.** If \( |t| > 1 \), the ray \( \Delta_{t,\theta} \) does not intersect the unit disc, and thus \( p_0(t) = 0 \). For \(|t| < 1\), the Radon transform is computed as the length of a cross section of a disc
\[ p_0(t) = 2\sqrt{1 - t^2}. \]

**Exercise 2.18.** We prove that the Gibbs oscillation amplitude is independent of the angle \( \theta \) and is equal to a one-dimensional Gibbs oscillation. Let us decompose \( f(x) \) into a continuous part \( f_0(x) \) and a discontinuity of constant amplitude \( A \):
\[ f(x) = f_0(x) + A u(\cos(\theta)x_1 + \sin(\theta)x_2) \]
where \( u(t) = 1_{[0, +\infty)}(t) \) is the one-dimensional Heaviside function. The filter satisfies \( h_\zeta(x_1, x_2) = g_\zeta(x_1) g_\zeta(x_2) \) with \( g_\zeta(t) = \sin(\xi t)/(\pi t) \). The Gibbs phenomena is produced by the discontinuity corresponding to the Heaviside function so we can consider that \( f_0 = 0 \). Let us suppose that \( |\theta| \leq \pi/4 \), with no loss of generality. We first prove that
\[ f \star h_\zeta(x) = f \star g_\zeta(x) \tag{1} \]
where \( \hat{g}_\zeta(\omega_1, \omega_2) = 1_{[-\xi, \xi]}(\omega_2) \). Indeed \( f(x) \) is constant along any line of angle \( \theta \), one can thus verify that its Fourier transform has a support located on the line in the Fourier plane, of angle \( \theta + \pi/2 \) which goes through 0. It results that \( f(\omega) \hat{h}_\zeta(\omega) = f(\omega) \hat{g}_\zeta(\omega) \) because the filtering limits the support of \( \hat{f} \) to \( \omega_2 \leq \zeta \). But \( g_\zeta(x_1, x_2) = \delta(x_1) \sin(\xi x_2)/(\pi x_2) \). The convolution (1) is thus a one-dimensional convolution along the \( x_2 \) variable, which is computed in the Gibbs Theorem 2.8. The resulting one-dimensional Gibbs oscillations are of the order of \( A \times 0.045 \).